gene has an affinity for the cytogene, this lag period is the time required for transfer of the cytogene from the chromogene to the cytoplasm and the development of a measurable amount of CO_2. According to this view, the chromogenes are occupied by cytogenes at those times when substrate is absent from the cell. When the substrate appears, the diffusion gradient toward it robs the chromogene of most of its cytogenes. When the substrate has been transformed, the cytogenes return to the locus. If the substrate is one rarely encountered, the stored cytogenes may be called forth only rarely. The cytogenes diffuse from the chromogene into the cytoplasm where they transform a specific substrate and duplicate themselves at the same time. After all the substrate has been transformed, a few molecules return to the chromogene and the excess of cytogenes is converted into other similar enzymes. Plasmagenes or viruses are modified cytogenes which can be transmitted without recourse to a chromosome locus.

Simple "loss" mutations may be the result of either (1) transforming the chromogene into a site which no longer has any affinity for the cytogene, or (2) complete destruction or loss of the cytogene. Some hypomorphic mutations may be changes in the locus which reduce the affinity of the chromogene for the cytogene. Other mutations may be alterations in the structure of the cytogene or simultaneous alteration in chromogene and cytogene.

[1] Lindegren, C. C., *Ann. Mo. Bot. Gard.*, **32**, 107–123 (1945).

[2] Lindegren, C. C., Spiegelman, S., and Lindegren, G., these PROCEEDINGS, **30**, 346–352 (1944).

[3] Sonneborn, T. M., *Ann. Mo. Bot. Gard.*, **32**, 213–221 (1945).

[4] Spiegelman, S., Lindegren, C. C., and Lindegren, G., these PROCEEDINGS, **31**, 95–102 (1945).

AN EXTENSION OF SCHUSTER'S INTEGRAL

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1. Schuster's integral, which occurs in the theory of total reflection of light,[1] is

$$\int_0^\infty (C^2 + S^2)dx = (\pi/8)^{1/2}, \qquad C = \int_x^\infty \cos t^2 dt, \qquad S = \int_x^\infty \sin t^2 dt$$

where the notation is that used by Nielsen[2] and Hardy.[3] An extension is obtained by modifying the analysis of Ingham.[4] If

$$C(x) = \int_x^\infty \cos (t^n)dt, \qquad S(x) = \int_x^\infty \sin (t^n)dt \qquad (n > 1)$$

the extension of Ingham's lemma is

$$C(x) = (l/n) \int_x^\infty (dt/t)(d/dt)(\sin t^n)$$
$$= -(1/nx) \sin (x^n) + (1/n) \int_x^\infty \sin (t^n)dt/t^2 = 0(1/x)$$

when x is large and positive. It is easily seen also that $S(x) = 0(1/x)$ and so if

$$I = \int_0^\infty C(x)C(ax)x^{n-2}dx, \qquad J = \int_0^\infty S(x)S(ax)x^{n-2}dx \qquad (a > 0)$$

integration by parts gives

$$n(n - 1)I = n\int_0^\infty x^{n-1}dx[\cos (x^n)C(ax) + a \cos (a^nx^n)C(x)]$$
$$= a\int_0^\infty \sin (x^n) \cos (a^nx^n)dx + a^{1-n} \int_0^\infty \sin (a^nx^n) \cos (x^n)dx$$
$$= {}^1/_2a^{1-n}S(0)(s^m - d^m), \qquad S = 1 + a^n, \qquad d = |1 - a^n|, \quad m = 1 - 1/n$$

$$n(n - 1)J = n\int_0^\infty x^{n-1}dx[\sin (x^n)S(ax) + a \sin (a^nx^n)S(x)]$$
$$= (1 + a^{1-n})S(0) - a\int_0^\infty \cos (x^n) \sin (a^nx^n)dx - a^{1-n} \int_0^\infty \times \cos (a^nx^n) \sin (x^n)dx$$
$$= {}^1/_2a^{1-n}S(0)[2(1 + a^{n-1}) - s^m - d^m], \qquad S(0) = \frac{1}{n}\Gamma\left(\frac{1}{n}\right) \times \sin (\pi/2n).$$

In particular, when $n = 2$

$$I = (^1/_4a)[(1 + a^2)^{1/2} - |1 - a^2|^{1/2}]S(0)$$
$$J = (^1/_4a)[2(1 + a) - (1 + a^2)^{1/2} - |1 - a^2|^{1/2}]S(0)$$
$$I + J = (^1/_2a)S(0)[1 + a - |1 - a^2|^{1/2}]$$
$$J - I = (^1/_2a)S(0)[1 + a - (1 + a^2)^{1/2}].$$

In these equations $C(0) = S(0) = (\pi/8)^{1/2}$ and so when $a = 1$ the third equation gives Schuster's relation. We also have

$$\int_0^\infty C(x)S(ax)dx = (^1/_4a)C(0)[2 \pm |a^2 - 1|^{1/2} - (a^2 + 1)^{1/2}]$$

where the upper or lower sign is taken according as a is greater or less than one. Similarly,

$$\int_0^\infty S(x)C(ax)dx = (^1/_4a)C(0)[2a \mp |a^2 - 1|^{1/2} - (a^2 + 1)^{1/2}].$$

Returning to the more general case in which n does not have the special value 2 we note that if

$$K = \int_0^\infty C(x)S(ax)x^{n-2}dx \qquad (n > 1)$$

$$n(n - 1)K = {}^1/_2a^{1-n}C(0)[2 \pm d^m - s^m], \qquad C(0) = \frac{1}{n}\Gamma\left(\frac{1}{n}\right) \cos (\pi/2n).$$

2. A number of other integrals may be derived from the values of I, J, K with the aid of the relations

$$\left.\begin{aligned}\int_0^\infty a^{s-1}C(ax)da &= \frac{x^{-s}}{nz}\,\Gamma\left(\frac{1+z}{n}\right)\cos\left[\frac{\pi(1+z)}{2n}\right]\\ \int_0^\infty a^{s-1}S(ax)da &= \frac{x^{-s}}{nz}\,\Gamma\left(\frac{1+z}{n}\right)\sin\left[\frac{\pi(1+z)}{2n}\right]\end{aligned}\right\}\quad (z > -1).$$

Thus from I we obtain the relation

$$\int_0^\infty t^{z-1}[(1+t)^p - |1-t|^p]dt = -\frac{p}{z\Gamma(2-p)}\sec\left(\tfrac{1}{2}p\pi\right)\qquad (-1 < z < 1-p).$$

3. We next write for brevity

$$\int_x^\infty e^{-t^n}dt = E(x)\qquad (n > 1).$$

Then, if

$$L = \int_0^\infty x^{n-2}E(x)E(ax)dx$$

we find on integration by parts that

$$n(n-1)L = a^{1-n}[1 + a^{n-1} - (1+a^n)^{1-1/n}]E(0).$$

The transformation $u = t^n$ indicates that $E(0) = \Gamma(1 + 1/n)$

$$E(x) = (1/n)\Gamma(1/n, x^n)$$

consequently, the equation may be written in the form

$$\int_0^\infty \Gamma(m, u)\Gamma(m, xu)du/u^m = \Gamma(m-1)[(1+1/x)^{m-1} - 1 - x^{m-1}]\qquad (0 < m < 1).$$

In the particular case in which $n = 2$

$$\int_0^\infty E(x)E(ax)dx = (1/a)[1 + a - (1+a^2)^{1/2}]E(0)$$

and $E(0) = \frac{1}{2}\pi$.

[1] Schuster, Sir Arthur, *Proc. Roy. Soc. London*, **107A**, 15–30 (1925).

[2] Nielsen, N., *Theorie des Integrallogarithmus und verwandter Transzendenten*, Leipzig, 1906.

[3] Hardy, G. H., *Proc. London Math. Soc.*, **24** (2), Records for Feb. 12 (1925). In an earlier paper, **9** (2), 126–144 (1911), a discussion is given of the convergence of integrals of type $\int_0^\infty \exp(ix^a)dx/x^b$.

[4] Ingham, A. E., *Jour. Lond. Math. Soc.*, **1**, 34–35 (1926).